

REMARKS ON THE CONTINUOUS L^2 -COHOMOLOGY FOR VON NEUMANN ALGEBRAS

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ABSTRACT. We prove that norm continuous derivations from a von Neumann algebra into the algebra of operators affiliated with its tensor square are automatically continuous for the strong operator topology as well. From this we derive previously known computational results regarding the first continuous L^2 -Betti number for von Neumann algebras, and furthermore prove that it vanishes whenever the von Neumann algebra in question is a property (T) factor.

1. INTRODUCTION

The theory of L^2 -Betti numbers has been generalized to a vast number of different contexts since the seminal work of Atiyah [Ati76]. One recent such generalization is due to Connes and Shlyakhtenko [CS05] who introduced L^2 -Betti numbers for subalgebras in finite von Neumann algebras, with the main purpose being to obtain a suitable notion of L^2 -Betti numbers for arbitrary II_1 -factors. Although their definitions are very natural and generalize nicely classical results from group theory, it has proven to be quite difficult to perform concrete calculations. The most advanced computational result so far is due to Thom [Tho08] who proved that the L^2 -Betti numbers vanish for von Neumann algebras with diffuse center. Notably, the problem of computing a positive degree L^2 -Betti number for a single II_1 -factor has been open since the birth of the theory! Due to this evident drawback, Thom [Tho08] introduced a continuous version of the first L^2 -Betti number, which turns out to be much more manageable than its algebraic counterpart. The first continuous L^2 -Betti number is defined as the von Neumann dimension of the first continuous Hochschild cohomology of the von Neumann algebra M with values in the algebra of operators affiliated with $M \bar{\otimes} M^{\text{op}}$. The word “continuous” here means that we restrict attention to those derivations that are continuous from the norm topology on M to the measure topology on the affiliated operators.

In this paper we continue the study of Thom’s continuous version of the first L^2 -Betti numbers and our first result shows that norm continuous derivations are automatically continuous for the strong operator topology as well. This allows us to derive all previously known vanishing results concerning the first continuous L^2 -Betti number, and furthermore to prove that it vanishes for II_1 factors with property (T). Lastly, we give a short new cohomological proof of the well-known fact that the (“non-continuous”) first L^2 -Betti number vanishes for von Neumann algebras with diffuse center.

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2. PRELIMINARIES

In this section we briefly recapitulate the theory of non-commutative integration and the theory of L^2 -Betti numbers for von Neumann algebras.

2.1. Non-commutative integration. Let us recall some facts from the theory of non-commutative integration, cf. [Nel74], [Tak03, IX.2]. Let N be a finite von Neumann algebra equipped with a normal, faithful, tracial state τ . Consider N in its representation on the GNS-space arising from τ , and let \mathcal{N} be the algebra of (potentially) unbounded, closed, densely defined operators affiliated with N . We equip \mathcal{N} with the *measure topology*, defined by the following two-parameter family of neighbourhoods of zero:

$$N(\varepsilon, \delta) = \{a \in \mathcal{N} \mid \exists p \in \text{Proj}(N) : \|ap\| < \varepsilon, \tau(p^\perp) < \delta\}, \quad \varepsilon, \delta > 0.$$

With this topology, \mathcal{N} is a complete metrizable topological vector space and the multiplication map

$$(a, b) \mapsto ab : \mathcal{N} \times \mathcal{N} \rightarrow \mathcal{N}$$

is uniformly continuous when restricted to products of bounded subsets [Nel74, Theorem 1]. Convergence with respect to the measure topology is also referred to as convergence *in measure*. We also introduce the notation

$$N(0, \delta) = \{a \in \mathcal{N} \mid \exists p \in \text{Proj}(N) : ap = 0, \tau(p^\perp) < \delta\}.$$

and

$$N(\varepsilon, 0) = \{a \in \mathcal{N} \mid \|a\| < \varepsilon\} \subset \mathcal{N}.$$

Notice that $N(0, \delta)$ and $N(\varepsilon, 0)$ are *not* zero neighbourhoods in the measure topology, but merely G_δ sets¹. However, the following additive and multiplicative properties continue to hold for all $\varepsilon_1, \varepsilon_2, \delta_1, \delta_2 \geq 0$, cf. [Nel74, Theorem 1]:

$$(2.1) \quad N(\varepsilon_1, \delta_1) + N(\varepsilon_2, \delta_2) \subset N(\varepsilon_1 + \varepsilon_2, \delta_1 + \delta_2),$$

$$(2.2) \quad N(\varepsilon_1, \delta_1) \cdot N(\varepsilon_2, \delta_2) \subset N(\varepsilon_1 \varepsilon_2, \delta_1 + \delta_2).$$

The noncommutative L^p -spaces $L^p(N, \tau)$ are naturally identified with subspaces of \mathcal{N} [Tak03, Theorem IX.2.13]. We fix the notation \xrightarrow{s} for strong convergence of elements in von Neumann algebras, $\xrightarrow{2}$ for the L^2 -convergence and \xrightarrow{m} for the convergence in measure of elements in \mathcal{N} . Clearly strong convergence implies convergence in 2-norm, and we remind the reader that for nets that are bounded in the operator norm the converse true — a fact we will use extensively in the sequel. As in the commutative case, the Chebyshev inequality is used to establish the following fact.

Lemma 2.1 ([Nel74, Theorem 5]). *For any $p \geq 1$ the inclusion $L^p(N, \tau) \subset \mathcal{N}$ is continuous; i.e. L^p -convergence implies convergence in measure.*

2.2. L^2 -Betti numbers for tracial algebras. In [CS05] Connes and Shlyakhtenko introduced L^2 -Betti numbers in the general setting of tracial $*$ -algebras; if M is a finite von Neumann algebra and $\mathcal{A} \subseteq M$ is any weakly dense unital $*$ -subalgebra its L^2 -Betti numbers are defined as

$$\beta_p^{(2)}(\mathcal{A}, \tau) = \dim_{M \overline{\otimes} M^{\text{op}}} \text{Tor}_p^{\mathcal{A} \odot \mathcal{A}^{\text{op}}}(M \overline{\otimes} M^{\text{op}}, \mathcal{A}).$$

Here the dimension function $\dim_{M \overline{\otimes} M^{\text{op}}}(-)$ is the extended von Neumann dimension due to Lück; we refer the reader to the monograph [Lüc02] for an extensive treatment of this notion. This definition is inspired by the well-known correspondence between representations of groups and bimodules over finite von Neumann algebras, and it extends the classical theory by means of the formula

¹But $N(0, \delta)$ is a zero-neighbourhood in the so-called rank metric.

$\beta_p^{(2)}(\Gamma) = \beta_p^{(2)}(\mathbb{C}\Gamma, \tau)$ whenever Γ is a discrete, countable group. In [Tho08] it is shown that the L^2 -Betti numbers also allow the following cohomological description:

$$\beta_p^{(2)}(\mathcal{A}, \tau) = \dim_{M \overline{\otimes} M^{\text{op}}} \text{Ext}_{\mathcal{A} \otimes \mathcal{A}}^p(\mathcal{A}, \mathcal{U}),$$

where \mathcal{U} denotes the algebra of operators affiliated with $M \overline{\otimes} M^{\text{op}}$. It is a classical fact [Lod98, 1.5.8] that the Ext-groups above are isomorphic to the Hochschild cohomology groups of \mathcal{A} with coefficients in \mathcal{U} , where the latter is considered as an \mathcal{A} -bimodule with respect to the actions

$$a \cdot \xi := (a \otimes 1)\xi \quad \text{and} \quad \xi \cdot b := (1 \otimes b^{\text{op}})\xi \quad \text{for } a, b \in \mathcal{A} \text{ and } \xi \in \mathcal{U}.$$

In particular the first L^2 -Betti number can be computed as the dimension of the right $M \overline{\otimes} M^{\text{op}}$ -module

$$H^1(\mathcal{A}, \mathcal{U}) = \frac{\text{Der}(\mathcal{A}, \mathcal{U})}{\text{Inn}(\mathcal{A}, \mathcal{U})}.$$

Here $\text{Der}(\mathcal{A}, \mathcal{U})$ denotes the space of derivations from \mathcal{A} to \mathcal{U} and $\text{Inn}(\mathcal{A}, \mathcal{U})$ denotes the space of inner derivations. We recall that a linear map δ from \mathcal{A} into an \mathcal{A} -bimodule \mathcal{X} is called a *derivation* if it satisfies

$$\delta(ab) = a \cdot \delta(b) + \delta(a) \cdot b \quad \text{for all } a, b \in \mathcal{A},$$

and that a derivation is called *inner* if there exists a vector $\xi \in \mathcal{X}$ such that

$$\delta(a) = a \cdot \xi - \xi \cdot a \quad \text{for all } a \in \mathcal{A}.$$

When the bimodule in question is \mathcal{U} with the above defined bimodule structure, the derivation property amounts to the following:

$$\delta(ab) = (a \otimes 1^{\text{op}})\delta(b) + (1 \otimes b^{\text{op}})\delta(a) \quad \text{for all } a, b \in \mathcal{A}.$$

Although the extended von Neumann dimension is generally not faithful, enlarging the coefficients from $M \overline{\otimes} M^{\text{op}}$ to \mathcal{U} has the effect that $\beta_1^{(2)}(\mathcal{A}, \tau) = 0$ if and only if $H^1(\mathcal{A}, \mathcal{U})$ vanishes [Tho08, Corollary 3.3 and Theorem 3.5]. In particular, in order to prove that $\beta_1^{(2)}(\mathcal{A}, \tau) = 0$ one has to prove that every derivation from \mathcal{A} into \mathcal{U} is inner.

These purely algebraically defined L^2 -Betti numbers have turned out extremely difficult to compute. Actually, the only computational result known (disregarding finite dimensional algebras) is that they vanish for von Neumann algebras with diffuse center (see [CS05, Corollary 3.5] and [Tho08, Theorem 2.2]). In particular, for II_1 -factors not a single computation of a positive degree L^2 -Betti is known, and furthermore this seems out of reach at the moment. It is therefore natural to consider variations of the definitions above that take into account the topological nature of M , and in [Tho08] Thom suggests to consider a first cohomology group consisting of (equivalence classes of) those derivations $\delta: \mathcal{A} \rightarrow \mathcal{U}$ that are closable from the norm topology to the measure topology. Note that when \mathcal{A} is norm closed these are exactly the derivations that are norm-measure topology continuous. We denote the space of closable derivations by $\text{Der}_c(\mathcal{A}, \mathcal{U})$, the continuous cohomology by $H_c^1(\mathcal{A}, \mathcal{U})$ and by $\eta_1^{(2)}(M, \tau)$ the corresponding continuous L^2 -Betti numbers; i.e.

$$\eta_1^{(2)}(\mathcal{A}, \tau) = \dim_{M \overline{\otimes} M^{\text{op}}} H_c^1(\mathcal{A}, \mathcal{U}).$$

These continuous L^2 -Betti numbers are much more manageable than their algebraic counterparts — for instance they are known [Tho08, Theorem 6.4] to vanish for von Neumann algebras that are non-prime and for those that contain a diffuse Cartan subalgebra.

Finally, let us fix a bit of notation. For the rest of this paper, we consider a finite von Neumann algebra M with separable predual. We endow M with a fixed faithful, normal, tracial state τ and consider M in the GNS representation on the Hilbert space $\mathcal{H} = L^2(M, \tau)$. The trace τ induces a faithful, normal, tracial state $\tau \otimes \tau$ on the von Neumann algebraic tensor product $M \overline{\otimes} M^{\text{op}}$ of M with its opposite algebra; abusing notation slightly, we will still denote it by τ . We will always consider the GNS representation of $M \overline{\otimes} M^{\text{op}}$ on $L^2(M \overline{\otimes} M^{\text{op}}, \tau)$ and let \mathcal{U} be the algebra of closed, densely defined, unbounded operators affiliated with $M \overline{\otimes} M^{\text{op}}$. Finally, we will use the symbol “ $\overline{\otimes}$ ” to denote tensor products of von Neumann algebras and “ \odot ” to denote algebraic tensor products, and, unless explicitly stated otherwise, all subalgebras in M are implicitly assumed to contain the unit of M .

3. STRONG CONTINUITY OF DERIVATIONS INTO AFFILIATED OPERATORS

In this section we prove that a derivation which is continuous for the norm topology is automatically continuous for the strong operator topology as well. Intuitively, this statement is based on the fact that strong convergence is “almost uniform”, which is known as the non-commutative Egorov theorem, cf. [Tak02, Theorem II.4.13]. The precise statement is as follows.

Theorem 3.1.

- i) *Let $A \subset M$ be a weakly dense C^* -algebra and let $\delta: A \rightarrow \mathcal{U}$ be a norm-measure topology continuous derivation. Then δ has a unique continuation to M .*
- ii) *Let $\delta: M \rightarrow \mathcal{U}$ be a norm-measure topology continuous derivation. Then δ is also continuous from the strong operator topology to the measure topology.*

Proof. By Kaplansky’s density theorem, the unit ball $(A)_1$ is strongly dense in $(M)_1$, and for $a \in (M)_1$ we may therefore choose a sequence² $a_n \in (A)_1$ with $a_n \xrightarrow{s} a$. Let us first prove that $\delta(a_n)$ is Cauchy in measure. So we fix an $\varepsilon > 0$ and want to find an N_0 such that $\delta(a_n) - \delta(a_m) \in N(\varepsilon, \varepsilon)$ for $\min\{m, n\} > N_0$. We first make use of the norm-measure topology continuity of $\delta: A \rightarrow \mathcal{U}$ to find a $\gamma > 0$ such that $\|a\| \leq \gamma$ implies that $\delta(a) \in N(\varepsilon/3, \varepsilon/3)$. Consider now $\mathbb{N} \times \mathbb{N}$ with the ordering

$$(m, n) \geq (m', n') \text{ iff } m \geq m' \text{ and } n \geq n',$$

and the net of self-adjoint elements $b_{(m,n)} := (a_m - a_n)^*(a_m - a_n)$. Then for every $\xi \in \mathcal{H}$ we have $\|b_{(m,n)}\xi\| \leq 2\|(a_m - a_n)\xi\|$ and hence $b_{(m,n)} \xrightarrow{s} 0$. Let now $f: \mathbb{R} \rightarrow [0, 1]$ be a continuous function with $f(x) = 1$ for $x \leq \gamma^2/4$ and $f(x) = 0$ for $x \geq \gamma^2$ and consider the net $h_{(m,n)} := f(b_{(m,n)})$. It follows that $\|h_{(m,n)}\| \leq 1$ and

$$\frac{\gamma^2}{4}(1 - h_{(m,n)}) \leq b_{(m,n)}.$$

Since the C^* -algebra generated by $b_{(m,n)}$ is commutative and both $b_{(m,n)}$ and $1 - h_{(m,n)}$ are positive this implies

$$0 \leq (1 - h_{(m,n)})^*(1 - h_{(m,n)}) \leq \frac{16}{\gamma^4} b_{(m,n)}^* b_{(m,n)},$$

and thus

$$\|1 - h_{(m,n)}\|_2 \leq \frac{4}{\gamma^2} \|b_{(m,n)}\|_2 \rightarrow 0.$$

Hence $1 - h_{(m,n)} \xrightarrow{2} 0$, and the convergence therefore holds in measure as well. Also note that $\|b_{(m,n)} h_{(m,n)}\| \leq \gamma^2$ and hence

²As the predual M_* is assumed separable, $(M)_1$ is a separable and metrizable space for the strong operator topology, and we can therefore do with sequences rather than nets.

$$(3.1) \quad \|(a_m - a_n)h_{(m,n)}\| \leq \gamma.$$

We now use the derivation property to obtain:

$$\begin{aligned} \delta(a_m) - \delta(a_n) &= \delta((a_m - a_n)h_{(m,n)}) + \delta((a_m - a_n)(1 - h_{(m,n)})) \\ &= \delta((a_m - a_n)h_{(m,n)}) + (1 \otimes (1 - h_{(m,n)})^{\text{op}})\delta(a_m - a_n) - \\ &\quad - ((a_m - a_n) \otimes 1^{\text{op}})\delta(h_{(m,n)}). \end{aligned}$$

By (3.1) and the choice of γ , the first summand is in $N(\varepsilon/3, \varepsilon/3)$. Let us now consider the second summand. The norm-measure topology boundedness of δ on A implies [Rud73, Thm. 1.32] that the set

$$\{\delta(a_m - a_n) \mid n, m \in \mathbb{N}\}$$

is bounded in \mathcal{U} . Hence it follows from the fact that $1 - h_{(m,n)} \xrightarrow{m} 0$ together with the uniform continuity of multiplication on bounded sets of \mathcal{U} , that there exists an N_1 such that $(1 \otimes (1 - h_{(m,n)})^{\text{op}})\delta(a_m - a_n) \in N(\varepsilon/3, \varepsilon/3)$ for $\min\{m, n\} > N_1$. Lastly we consider the third term. Again by norm-boundedness of δ , the set $\{\delta(h_{(m,n)})\}_{n,m \in \mathbb{N}}$ is bounded in \mathcal{U} . As a_n is strongly convergent it also converges in 2-norm; hence $a_n \otimes 1$ converges in 2-norm and is, in particular, a Cauchy sequence for the measure topology. Thus, there exists an N_2 such that

$$((a_m - a_n) \otimes 1^{\text{op}})\delta(h_{(m,n)}) \in N(\varepsilon/3, \varepsilon/3)$$

for $\min\{n, m\} > N_2$. Taking $N_0 = \max\{N_1, N_2\}$ establishes that $\delta(a_n)$ is a Cauchy sequence in the measure topology. Appealing to the completeness of \mathcal{U} , we may now define $\delta(a) := \lim_n \delta(a_n)$; it is routine to check that this yields a well-defined derivation $\delta: M \rightarrow \mathcal{U}$.

If $\delta: M \rightarrow \mathcal{U}$ is a derivation, then by repeating the above arguments for $m = \infty$, $a = a_\infty$, we get strong continuity of δ on bounded sets: if $a_n \xrightarrow{s} a$ and the sequence a_n is uniformly bounded, then $\delta(a_n) \xrightarrow{m} \delta(a)$. As the strong operator topology is metrizable on bounded sets, this continues to hold for bounded nets instead of sequences. To prove strong continuity in general, we take a (possibly unbounded) net a_λ such that $a_\lambda \xrightarrow{s} 0$. Define $A_\lambda := W^*(a_\lambda^* a_\lambda, 1)$. This is a commutative von Neumann algebra and we can therefore find a measure space (X_λ, μ_λ) and an isomorphism $\alpha_\lambda: A_\lambda \rightarrow L^\infty(X_\lambda)$ such that

$$\tau(a) = \int_{X_\lambda} \alpha_\lambda(a) d\mu_\lambda \quad \text{for all } a \in A_\lambda.$$

Now define $p_\lambda \in A_\lambda$ to be the pull-back via α_λ of the characteristic function corresponding to the set

$$\{t \in X_\lambda \mid \alpha_\lambda(x_\lambda^* x_\lambda)(t) \leq 1\}.$$

Then $\|a_\lambda p_\lambda\|^2 = \|p_\lambda a_\lambda^* a_\lambda p_\lambda\| \leq \|a_\lambda^* a_\lambda p_\lambda\| \leq 1$. Using the Chebyshev inequality, we furthermore get

$$\tau(1 - p_\lambda) = \mu_\lambda(\{t \in X_\lambda \mid \alpha_\lambda(x_\lambda^* x_\lambda)(t) \geq 1\}) \leq \int_{X_\lambda} \alpha_\lambda(a_\lambda^* a_\lambda) d\mu_\lambda = \|a_\lambda\|_2^2 \rightarrow 0.$$

Thus, p_λ is a norm bounded net that converges to 1 strongly and is bounded and, by what was just proven, it follows that $\delta(p_\lambda) \xrightarrow{m} \delta(1) = 0$. Now we use the derivation property of δ :

$$(3.2) \quad \delta(a_\lambda) = \delta(a_\lambda p_\lambda) + (1 \otimes (1 - p_\lambda)^{\text{op}})\delta(a_\lambda) - (a_\lambda \otimes 1^{\text{op}})\delta(p_\lambda).$$

First note that since strong convergence implies L^2 -convergence we have

$$\|a_\lambda p_\lambda\|_2 = \|J p_\lambda J(a_\lambda)\|_2 \leq \|a_\lambda\|_2 \rightarrow 0,$$

and since $\|a_\lambda p_\lambda\| \leq 1$ we get $a_\lambda p_\lambda \xrightarrow{s} 0$. Since we already know that δ respects bounded strong convergence, the first summand in (3.2) converges to 0 in measure. The second summand converges to 0 in measure because $1 - p_\lambda \in N(0, \varepsilon_\lambda)$ with $\varepsilon_\lambda = 1 - \tau(p_\lambda) \rightarrow 0$ and $\delta(a_\lambda) \in N(\gamma_\lambda, \varepsilon_\lambda)$ for some $\gamma_\lambda > 0$ [Tak03, Lemma IX.2.3]. The third summand converges to 0 in measure because $\delta(p_\lambda) \xrightarrow{m} 0$ and multiplication is measure continuous. This finishes the proof. \square

Remark 3.2. Bearing in mind the numerous automatic continuity results for derivations between operator algebras (see [SS95] and references therein), it is of course natural to ask if norm continuity of a derivation $\delta: M \rightarrow \mathcal{U}$ is also automatic. We were not able to prove this, and it seems to be a difficult question to answer. One reason being the absence of examples of finite von Neumann algebras (or C^* -algebras, for that matter) for which a non-inner derivation into the algebra \mathcal{U} is known to exist. Moreover, the fact that we are considering the operators affiliated with $M \overline{\otimes} M^{\text{op}}$ (as opposed to M itself) has to play a role if automatic norm continuity is to be proven, as there are examples of derivations from M into the operators affiliated with M which are not norm-measure topology bounded. This follows from [BCS06] where the authors exhibit a (commutative) finite von Neumann algebra M for which there exists a derivation $\delta: \mathcal{U}(M) \rightarrow \mathcal{U}(M)$ which is not measure-measure continuous. If the restriction $\delta|_M$ were norm-measure continuous, then it is not difficult to see that the graph of the original derivation δ is closed (in the product of the measure topologies), and hence it cannot be discontinuous. If norm continuity turns out to be automatic, there is of course no difference between the ordinary and the continuous L^2 -Betti numbers, and one might even take the standpoint that if there is no such automatic continuity, then continuity has to be imposed in order to get a satisfactory theory.

4. THE FIRST CONTINUOUS L^2 -BETTI NUMBER

In this section we apply the above automatic continuity result to obtain information about the first continuous L^2 -Betti number for von Neumann algebras. Some of the results presented are already known, or implicit in the literature, but since the proofs are new and quite simple we hope to shed new light on these results. The main new result in this section is Theorem 4.9 which shows that the first continuous L^2 -Betti number vanishes for property (T) factors.

Recall that if $N \subseteq M$ is an inclusion of von Neumann algebras, then the normalizer of N in M is defined as the set of unitaries in M which normalize N :

$$\mathcal{N}_M(N) = \{u \in U(M) \mid u^* N u = N\}.$$

The following lemma appeared in [Tho08], but we include its proof for the sake of completeness.

Lemma 4.1 ([Tho08, Lemma 6.5]). *Let $\delta: M \rightarrow \mathcal{U}$ be a derivation which vanishes on a diffuse subalgebra $N \subset M$ and let $u \in \mathcal{N}_M(N)$. Then $\delta(u) = 0$.*

Proof. Let $h \in N$ be a diffuse element. Since $\delta(u^*) = -(u^* \otimes u^{\text{op}})\delta(u)$ we get

$$\begin{aligned} 0 &= \delta(uhu^*) = (1 \otimes (hu^*)^{\text{op}})\delta(u) + (uh \otimes 1^{\text{op}})\delta(u^*) \\ &= (1 \otimes (hu^*)^{\text{op}})\delta(u) - (uhu^* \otimes u^{*\text{op}})\delta(u) \\ &= (u \otimes u^{*\text{op}})(1 \otimes h^{\text{op}} - h \otimes 1^{\text{op}})(u^* \otimes 1)\delta(u). \end{aligned}$$

Since h is diffuse, $1 \otimes h^{\text{op}} - h \otimes 1$ is not a zero divisor in \mathcal{U} , and it follows that $\delta(u) = 0$. \square

The next lemma, which might be of independent interest, shows that if a von Neumann algebra only allows inner derivations into the algebra of operators affiliated with its double, then the same is true for the algebra of operators affiliated with the double of any other von Neumann algebra containing it.

Lemma 4.2. *Let $N \subseteq M$ be a sub-von Neumann algebra. Then the following statements are equivalent:*

- i) $\beta_1^{(2)}(N, \tau) = 0$,
- ii) *every derivation $\delta: N \rightarrow \mathcal{U}$ is inner (where \mathcal{U} is considered as an N -bimodule via the inclusion $N \subseteq M$).*

Proof. By [Lod98, 1.5.9] we have $H^1(N, \mathcal{U}) = \text{Ext}_{N \odot N^{\text{op}}}^1(N, \mathcal{U})$ and by [Tho08, Theorem 3.5] this right \mathcal{U} -module is isomorphic to

$$\text{Hom}_{\mathcal{U}}(\text{Tor}_1^{N \odot N^{\text{op}}}(\mathcal{U}, N), \mathcal{U}).$$

Furthermore, by [Tho08, Corollary 3.3] this module vanishes exactly when

$$\dim_{M \overline{\otimes} M^{\text{op}}} \text{Tor}_1^{N \odot N^{\text{op}}}(\mathcal{U}, N) = 0.$$

But since $\mathcal{U} \odot_{M \overline{\otimes} M^{\text{op}}} -$ and $M \overline{\otimes} M^{\text{op}} \odot_{N \overline{\otimes} N^{\text{op}}} -$ are both flat and dimension preserving (see [Rei01, Proposition 2.1 and Theorem 3.11] and [Lüc02, Theorem 6.29]) we get

$$\begin{aligned} \dim_{M \overline{\otimes} M^{\text{op}}} \text{Tor}_1^{N \odot N^{\text{op}}}(\mathcal{U}, N) &= \dim_{M \overline{\otimes} M^{\text{op}}} \text{Tor}_1^{N \odot N^{\text{op}}}(\mathcal{U} \odot_{M \overline{\otimes} M^{\text{op}}} M \overline{\otimes} M^{\text{op}}, N) \\ &= \dim_{M \overline{\otimes} M^{\text{op}}} \mathcal{U} \odot_{M \overline{\otimes} M^{\text{op}}} \text{Tor}_1^{N \odot N^{\text{op}}}(M \overline{\otimes} M^{\text{op}}, N) \\ &= \dim_{M \overline{\otimes} M^{\text{op}}} \text{Tor}_1^{N \odot N^{\text{op}}}(M \overline{\otimes} M^{\text{op}}, N) \\ &= \dim_{M \overline{\otimes} M^{\text{op}}} \text{Tor}_1^{N \odot N^{\text{op}}}(M \overline{\otimes} M^{\text{op}} \odot_{N \overline{\otimes} N^{\text{op}}} N \overline{\otimes} N^{\text{op}}, N) \\ &= \dim_{M \overline{\otimes} M^{\text{op}}} M \overline{\otimes} M^{\text{op}} \odot_{N \overline{\otimes} N^{\text{op}}} \text{Tor}_1^{N \odot N^{\text{op}}}(N \overline{\otimes} N^{\text{op}}, N) \\ &= \dim_{N \overline{\otimes} N^{\text{op}}} \text{Tor}_1^{N \odot N^{\text{op}}}(N \overline{\otimes} N^{\text{op}}, N) \\ &= \beta_1^{(2)}(N, \tau). \end{aligned}$$

This finishes the proof. \square

4.1. The case of diffuse center. Using methods from free probability, Connes and Shlyakhtenko proved in [CS05, Corollary 3.5] that $\beta_1^{(2)}(M, \tau) = 0$ when M has diffuse center, and using homological algebraic methods this was generalized by Thom to higher L^2 -Betti numbers in [Tho08, Theorem 2.2]. In this section we give a short cohomological proof of this result in degree one.

Theorem 4.3 ([CS05, Corollary 3.5]). *If M has diffuse center then every derivation $\delta: M \rightarrow \mathcal{U}$ is norm-measure topology continuous and $\beta_1^{(2)}(M, \tau)$ is zero.*

Proof. Since the center $Z(M)$ is diffuse we can choose an identification $Z(M) = L^\infty(\mathbb{T}) = L\mathbb{Z}$. Denote by h a diffuse, selfadjoint generator of $Z(M)$. To see that δ is bounded, it suffices to prove that its graph is closed [Rud73, Theorem 2.15]; let therefore $x_n \in M$ with $\|x_n\| \rightarrow 0$ and $\delta(x_n) \xrightarrow{m} \eta$. Then

$$\begin{aligned} 0 = \delta([x_n, h]) &= (x_n \otimes 1^{\text{op}})\delta(h) + (1 \otimes h^{\text{op}})\delta(x_n) - (1 \otimes x_n^{\text{op}})\delta(h) - (h \otimes 1^{\text{op}})\delta(x_n) \\ &\xrightarrow{m} (1 \otimes h^{\text{op}} - h \otimes 1)\eta, \end{aligned}$$

and since h is diffuse $(1 \otimes h^{\text{op}} - h \otimes 1)$ is not a zero-divisor in \mathcal{U} ; hence $\eta = 0$. We now claim that δ has to be inner on $Z(M) = L\mathbb{Z}$. If this were not the case, then, by Lemma 4.2, there exists a non-inner derivation $\delta': L\mathbb{Z} \rightarrow \mathcal{U}(L\mathbb{Z} \overline{\otimes} L\mathbb{Z})$. By what was just proven, δ' is norm-measure continuous and therefore, by Theorem 3.1, also

strong-operator continuous. Because of this, the restriction of δ' to the complex group algebra $\mathbb{C}[Z]$ has to be non-inner, contradicting the fact that $0 = \beta_1^{(2)}(\mathbb{Z}) = \beta_1^{(2)}(\mathbb{C}[Z], \tau)$ (cf. [CS05, Proposition 2.3] and [CG86, Theorem 0.2]). Hence there exists $\xi \in \mathcal{U}$ such that δ agree with $\delta_\xi := [\cdot, \xi]$ on $Z(M)$. The difference $\delta - \delta_\xi$ therefore vanishes on $Z(M)$ and by Lemma 4.1 it has to vanish on every unitary in M . Since the unitaries span M linearly, we conclude that δ is globally inner. \square

Remark 4.4. By combining Lemma 4.1 and Theorem 4.3 with the automatic strong continuity from Theorem 3.1, one may at this point easily deduce the conclusion of [Tho08, Theorem 6.4]; namely that the first continuous L^2 -Betti number vanishes for non-prime von Neumann algebras as well as for von Neumann algebras admitting a diffuse Cartan subalgebra.

4.2. Comparison with dense $*$ -algebras. Everything stated in this subsection can also be derived from Thom's work in [Tho08], but since the results are not made explicit in the literature and follow easily from Theorem 3.1, we thought it worth while including them here.

Proposition 4.5. *Let $A \subseteq M$ be a weakly dense $*$ -subalgebra. Then*

$$\eta_1^{(2)}(M, \tau) \leq \eta_1^{(2)}(A, \tau) \leq \beta_1^{(2)}(A, \tau).$$

and if A is a C^ -algebra we have $\eta_1^{(2)}(M, \tau) = \eta_1^{(2)}(A, \tau)$.*

Proof. By Theorem 3.1 the map $H_c^1(M, \mathcal{U}) \rightarrow H_c^1(A, \mathcal{U})$ induced by restriction is injective in general and an isomorphism when A is a C^* -algebra. That $\eta_1^{(2)}(A, \tau) \leq \beta_1^{(2)}(A, \tau)$ is clear, as we have a natural injection $\text{Der}_c(A, \mathcal{U}) \rightarrow \text{Der}(A, \mathcal{U})$ for any algebra A . \square

To illustrate the usefulness of the above result, we record the following corollary.

Corollary 4.6.

- i) *If Γ is a discrete countable group with $\beta_1^{(2)}(\Gamma) = 0$ then also $\eta_1^{(2)}(L\Gamma, \tau) = 0$. In particular, the first continuous L^2 -Betti number of the hyperfinite factor R vanishes.*
- ii) *For the von Neumann algebra $L^\infty(O_n^+)$ associated with the free orthogonal quantum group O_n^+ we have $\eta_1^{(2)}(L^\infty(O_n^+), \tau) = 0$.*

Proof. By [CS05, Proposition 2.3], $\beta_1^{(2)}(\Gamma) = \beta_1^{(2)}(\text{C}\Gamma, \tau)$ and the first claim therefore follows from Proposition 4.5. The statement about the hyperfinite factor follows from this by realizing R as the group von Neumann algebra of an amenable i.c.c. group. Denoting by $\text{Pol}(O_n^+)$ the canonical dense Hopf $*$ -algebra in $L^\infty(O_n^+)$, it is known that $\beta_1^{(2)}(\text{Pol}(O_n^+), \tau) = 0$ (see [Kye08] for the case $n = 2$ and [Ver09] for the case $n \geq 3$) and hence, by Proposition 4.5, $\eta_1^{(2)}(L^\infty(O_n^+), \tau) = 0$. \square

Remark 4.7.

- 1) The result about the hyperfinite factor can also be obtained directly from the definition of hyperfiniteness by realizing R as the von Neumann algebraic direct limit of matrix algebras, for which it is also known [AK11, Example 6.9] that the (non-continuous) L^2 -Betti numbers of the corresponding algebraic direct limit vanishes.
- 2) It is known [BV97] that when Γ is a discrete, countable group with property (T) then $\beta_1^{(2)}(\Gamma) = 0$ and hence also $\eta_1(L\Gamma, \tau) = 0$. This indicates that it is reasonable to expect that $\eta_1(M, \tau) = 0$ for any property (T) factor M and we devote the next subsection to proving that this is indeed the case.

- 3) The von Neumann algebras $L^\infty(O_n^+)$ associated with the orthogonal quantum groups O_n^+ (see [Wan95, VDW96, Ban96] for definitions) behave in many ways like free group von Neumann algebras: they are solid (hence prime) II_1 -factors [VV07] with the Haagerup property [Bra11], and it is an open, and difficult, problem to determine if they are actually isomorphic to free group factors. Furthermore, it is natural to hope for a relation (perhaps even equality) between the L^2 -Betti numbers of a group and those of its associated von Neumann algebra, but at the moment such a result seems far out of reach. Bearing this, and the fact that $\beta_1^{(2)}(\mathbb{F}_n) = n - 1$, in mind it is therefore interesting to know that $\eta_1^{(2)}(L^\infty(O_n^+)) = 0$.

4.3. Factors with property (T). If Γ is a countable discrete group with property (T) it is well known that its first L^2 -Betti number vanishes. This observation goes back to the work of Gromov [Gro93], but the first complete proof was given by Bekka and Valette [BV97]. Applying recent results regarding the von Neumann dimension [TP11, Theorem 2.2], this can be easily deduced from the Delorme-Guichardet theorem (see e.g. [BdlHV08]), which characterizes property (T) of Γ in terms of vanishing of its first cohomology groups. The notion of property (T) for II_1 -factors was introduced by Connes and Jones in [CJ85] and in [Pet09] Peterson proved a version of the Delorme-Guichardet theorem in this context:

Theorem 4.8 ([Pet09, Theorem 0.1]). *Let M be a finite factor with separable predual. Then the following conditions are equivalent:*

- i) M has property (T);
- ii) M does not have property Γ and for any weakly dense $*$ -subalgebra $M_0 \subset M$ such that $1 \in M_0$ and M_0 contains a non- Γ set, every densely defined L^2 -closable derivation into a Hilbert N - N -bimodule whose domain contains M_0 is inner;
- iii) there exists a weakly dense $*$ -subalgebra $M_0 \subset M$ which is countably generated as a vector space and such that every densely defined L^2 -closable derivation into a Hilbert M - M -bimodule whose domain contains M_0 is inner.

In this section we apply Peterson's result to prove the following von Neumann algebraic version of the classical group theoretic result mentioned above.

Theorem 4.9. *Let M be a II_1 -factor with separable predual. If M has property (T), then $\eta_1^{(2)}(M, \tau) = 0$.*

Proof. Since M has property (T), we obtain from Theorem 4.8 a dense $*$ -subalgebra $M_0 \subset M$ such that M is countably generated as a vector space and such that any derivation from M_0 into a Hilbert M -bimodule H , which is closable as an unbounded operator from $L^2(M)$ to H , is inner. We first observe that

$$\dim_{M \overline{\otimes} M^{\text{op}}} \text{Der}_c(M, \mathcal{U}) = \dim_{M \overline{\otimes} M^{\text{op}}} \{\delta \in \text{Der}_c(M, \mathcal{U}) \mid \delta(M_0) \subseteq M \overline{\otimes} M^{\text{op}}\}.$$

To see this, it suffices by Sauer's local criterion [Sau05, Theorem 2.4] to prove that for each continuous derivation $\delta: M \rightarrow \mathcal{U}$ and each $\varepsilon > 0$ there exists a projection $p \in M \overline{\otimes} M^{\text{op}}$ such that $\tau(p^\perp) \leq \varepsilon$ and $\delta(-)p$ maps M_0 into $M \overline{\otimes} M^{\text{op}}$. Choose a countable basis $(e_n)_{n=1}^\infty$ for M_0 . Since each $\delta(e_n)$ is affiliated with $M \overline{\otimes} M^{\text{op}}$ there exists a projection $p_n \in M \overline{\otimes} M^{\text{op}}$ such that $\tau(p_n) \leq \frac{\varepsilon}{2^n}$ and such that $\delta(e_n)p_n \in M \overline{\otimes} M^{\text{op}}$. The projection $p := \bigwedge_n p_n$ therefore satisfies the requirements.

Now we have to prove that any derivation $\delta: M \rightarrow \mathcal{U}$ for which $\delta(M_0) \subseteq M \overline{\otimes} M^{\text{op}}$ is inner. We claim that it suffices to prove that δ is L^2 -closable from M_0 to

$L^2(M \overline{\otimes} M^{\text{op}})$. Indeed, if this is the case, then by Peterson's result there exists a vector $\xi \in L^2(M \overline{\otimes} M^{\text{op}})$ such that

$$\delta(a) = (a \otimes 1)\xi - (1 \otimes a^{\text{op}})\xi \quad \text{for all } a \in M_0.$$

Considering ξ as an operator in \mathcal{U} , we get that it implements δ on M_0 and hence by Theorem 3.1 it implements δ on all of M .

Thus, our task is to show that $\delta: M_0 \rightarrow L^2(M \overline{\otimes} M^{\text{op}})$ is L^2 -closable. Let $x_n \in M_0$ and assume that $x_n \xrightarrow{2} 0$ and $\delta(x_n) \xrightarrow{2} \eta$. Define $A_n := W^*(x_n^* x_n, 1)$. This is a commutative von Neumann algebra and we can therefore find a measure space (X_n, μ_n) and an isomorphism $\alpha_n: A_n \rightarrow L^\infty(X_n)$ such that

$$\tau(a) = \int_{X_n} \alpha_n(a) d\mu_n \quad \text{for all } a \in A_n.$$

Now define $p_n \in A_n$ to be the pull-back via α_n of the characteristic function corresponding to the set

$$\{t \in X_n \mid \alpha(x_n^* x_n)(t) \leq 1\}.$$

Then $\|x_n p_n\|^2 = \|p_n x_n^* x_n p_n\| \leq \|x_n^* x_n p_n\| \leq 1$ and using the Chebyshev inequality we furthermore get

$$\tau(1 - p_n) = \mu_n(\{t \in X_n \mid \alpha_n(x_n^* x_n)(t) \geq 1\}) \leq \int_{X_n} \alpha_n(x_n^* x_n) d\mu_n = \|x_n\|_2^2 \rightarrow 0.$$

Hence p_n converges to 1 strongly and by Theorem 3.1 it follows that $\delta(p_n) \xrightarrow{m} \delta(1) = 0$. Using the derivation property we now get

$$(4.1) \quad \delta(x_n p_n) = (1 \otimes p_n^{\text{op}}) \delta(x_n) + (x_n \otimes 1^{\text{op}}) \delta(p_n).$$

Since $\delta(x_n) \xrightarrow{2} \eta$ and $1 \otimes p_n^{\text{op}} \xrightarrow{2} 1 \otimes 1^{\text{op}}$ the convergence also holds in measure and hence $(1 \otimes p_n^{\text{op}}) \delta(x_n) \xrightarrow{m} \eta$. Moreover, since $x_n \otimes 1 \xrightarrow{2} 0$ the convergence also holds in measure and as $\delta(p_n) \xrightarrow{m} 0$ we have $(x_n \otimes 1) \delta(p_n) \xrightarrow{m} 0$. Hence the right hand side of (4.1) converges in measure to η . Looking at the left hand side, we have $\|x_n p_n\| \leq 1$ and that

$$\|x_n p_n\|_2 = \|J p_n J(x_n)\|_2 \leq \|x_n\|_2 \rightarrow 0.$$

Hence $x_n p_n \xrightarrow{s} 0$ and therefore $\delta(x_n p_n) \xrightarrow{m} 0$. As the measure topology is Hausdorff we conclude that $\eta = 0$, and hence that δ is L^2 -closable. \square

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